

The Physics of Granular Mechanics

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Abstract

The *hydrodynamic* approach to a continuum mechanical description of granular behavior is reviewed and elucidated. By considering energy and momentum conservation simultaneously, the general formalism of *hydrodynamics* provides a systematic method to derive the structure of constitutive relations, including all gradient terms needed for nonuniform systems. An important input to arrive at different relations (say, for Newtonian fluid, solid and granular medium) is the energy, especially the number and types of its variables.

Starting from a careful examination of the physics underlying granular behavior, we identify the independent variables and suggest a simple and qualitatively appropriate expression for the granular energy. The resultant hydrodynamic theory, especially the constitutive relation, is presented and given preliminary validation.

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I. INTRODUCTION

When unperturbed, sand piles persist forever, demonstrating in plain sight granular media’s ability to sustain shear stresses – an ability that is frequently considered the defining property of solids. On the other hand, when tapped, the same pile quickly degrades, to form a layer (possibly a monolayer) of grains minimizing the gravitational energy. This is typical of liquids. The microscopic reason for this dichotomy is clear: The grains are individually (and ever so slightly) deformed if buried in a pile, which is what sustains the shear stress. When tapped, the grains jiggle and shake, and briefly lose contact with one another. This is why they get rid of some of their deformation – which shows up, macroscopically, as a gradual loss of the static shear stress and a continual flattening of the pile. [42]

When sand is being sheared at a constant rate, both solid and fluid behavior are operative. First, the grains are being deformed, increasing the shear stress as any solid would. Second, the same shear rate also provokes some jiggling, just as if the grains were lightly tapped. [43] This leads to a fluid-like relaxation of the shear stress – the larger the shear rate, the stronger the jiggling, and the quicker the relaxation. Note the reason why loading and unloading give different responses (called *incremental nonlinearity* [1, 2]): When being loaded, the solid part of granular behavior increases the stress, while the fluid part decreases it. During unloading, both work in the same direction to reduce the stress.

This entangled behavior, we suspect, lies at the heart of the difficulty modeling sand macroscopically. In addition, there is a “history-dependence” of granular behavior that, being experimentally obvious but conceptually confused and ill-defined, further perplexes the modeler. Obviously, if sand can be characterized, as do other systems, by a *complete set of state variables*, any history-dependence only indicates that the experiments were run at different values of these variables. This is what we believe happens.

The *hydrodynamic theory* is a powerful approach to continuum-mechanical description (or macroscopic field theory), pioneered by Landau [3] and Khalatnikov [4] in the context of superfluid helium. By considering energy and momentum conservation simultaneously, and combining both with thermodynamic considerations, this approach is capable of cogently deducing, among others, the proper constitutive relation. Hydrodynamics [5] has since been successfully employed to account for many condensed systems, including liquid crystals [6, 7], superfluid ^3He [8–10], superconductors [11–13], macroscopic electro-magnetism [14–

16] and ferrofluids [17–20]. Transiently elastic media such as polymers are under active consideration at present [21–23].

Two steps are involved in deriving the theory hydrodynamically, the first specifies the theory’s *structure*: Being a function of the state variables, the energy itself is not independent. Nevertheless, the form of the energy density $w(s, \rho)$ is left unspecified in this first step, and the differential equations are given in terms of the energy density w , its variables and conjugate variables. [Conjugate variables are the derivatives of the energy with respect to the variables, say temperature $T(s, \rho) \equiv \partial w / \partial s$ and chemical potential $\mu(s, \rho) \equiv \partial w / \partial \rho$ for s, ρ , the entropy and mass density]. In a continuum theory, a number of transport coefficients [such as the viscosity $\eta(s, \rho)$ or the heat diffusion coefficient $\kappa(s, \rho)$] are needed to parameterize dissipation and entropy production. Neither is their functional dependence specified.

A theory is unique and useful, of course, only when its energy and transport coefficients are made specific, in a second step. This division is sensible, because the first step is systematic, the second is not. The first starts with clearly spelt-out assumptions based on the basic physics of the system at hand, which is followed by a derivation that is algebraic in nature, and hence rather cogent. The second step is a fitting process – one looks for appropriate expressions, by trial and error, for a few scalar functions that, when embedded into the structure of the theory, will yield satisfactory agreement with the many experimental data.

Starting from the physics of granular deformation and its depletion by jiggling, we have identified the variables and derived the structure of the equations governing their temporal evolution [24], calling it GSH, for granular solid hydrodynamics. But our second step is not yet complete, and some proposed functional dependencies are still tentative. The expression for the energy appears quite satisfactory, but our notion of the transport coefficients is still vague. Our final goal is a transparent theory with a healthy mathematical structure that is capable of modeling sand in its full width of behavior, from static stress distribution, via elastoplastic deformation [25, 26], to granular flow property at higher velocities [27–30].

II. GRANULAR STATE VARIABLES

In this section, we determine the complete set of granular variables starting from the elementary physics of granular deformation and its depletion by jiggling.

A. The Elastic Strain

If a granular medium is sheared, the grains jiggle, roll and slide, in addition to being deformed. Only the latter leads to a reversible energy storage. Therefore, the strain $\varepsilon_{ij} = u_{ij} + p_{ij}$ has two parts, the elastic and plastic one, with the first *defined* as the part that changes the energy. Hence the energy density $w(u_{ij})$ is a function of the elastic strain u_{ij} , which alone we identify as a state variable. For analogy, think of riding a bike on a snowy path, up a steep slope. The rotation of the wheel, containing slip and center-of-mass motion, corresponds to the total displacement d . The gravitational energy $w(d_t)$ of the cyclist and his bike depends only on the center-of-mass movement d_t , the “elastic” or energy-changing portion here. And the gravitational force on the center of mass is $f_g = -\partial w/\partial d_t$. Similarly, the elastic stress is $\pi_{ij} = -\partial w(u_{ij})/\partial u_{ij}$. When grains jiggle, granular deformation relax, hence

$$\partial_t u_{ij} = v_{ij} - u_{ij}/\tau, \quad (1)$$

with the usual elastic term $v_{ij} \equiv \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$, and a relaxation term $-u_{ij}/\tau$ that accounts for plasticity. [Note because the total strain obeys $\partial_t \varepsilon_{ij} = v_{ij}$, the evolution of the plastic strain $p_{ij} \equiv \varepsilon_{ij} - u_{ij}$ is also fixed by Eq (1), and given as $\partial_t p_{ij} = u_{ij}/\tau$.] To understand how plasticity comes about, consider first the following scenario with $\tau = \text{constant}$. If a granular medium is deformed quickly enough by an external force, leaving little time for relaxation, $\int (u_{ij}/\tau) dt \approx 0$, we have $u_{ij} \approx \varepsilon_{ij} = \int v_{ij} dt$ and $p_{ij} = 0$ right after the deformation. The built-up in elastic energy and stress π_{ij} is maximal. If released at this point, the system snaps back toward its initial state, as prescribed by momentum conservation, $\partial_t (\rho v_i) + \nabla_j \pi_{ij} = 0$, displaying an elastic, reversible behavior. But if the system is being held still ($\partial_t \varepsilon_{ij} = v_{ij} = 0$) long enough, the elastic strain u_{ij} will relax, $\partial_t u_{ij} = -u_{ij}/\tau$, while the plastic strain grows accordingly, $\partial_t p_{ij} = u_{ij}/\tau$. When u_{ij} vanishes, elastic energy $w(u_{ij})$ and stress π_{ij} are also gone, implying $\partial_t (\rho v_i) = 0$. The system now stays where it is when released, and no longer returns to its original position. This is what we call plasticity.

However, $1/\tau$ is not a constant in sand: It grows with the jiggling of the grains (as the deformation is lost more quickly) and vanishes if they are at rest. If we quantify the jiggling by the associated kinetic energy, or (via the gas analogy) by a granular temperature T_g , we could account for this by assuming $1/\tau \sim T_g$.

As discussed above, a shear rate would jiggle the grains, giving rise to T_g . For a constant rate, an expression of the form $T_g \sim \sqrt{v_{ij}v_{ij}} \equiv ||v_s||$ is appropriate [see Eq (13) below]. Inserting $1/\tau = \Lambda ||v_s||$ (with Λ the proportionality coefficient) into Eq (1), we obtain the rate-independent expression, $\partial_t u_{ij} = v_{ij} - \Lambda u_{ij} ||v_s||$. Being a function of u_{ij} , the stress $\pi_{ij}(u_{ij})$ therefore obeys the evolution equation,

$$\partial_t \pi_{kl} = M_{klij} \partial_t u_{ij} = M_{klij} (v_{ij} - \Lambda u_{ij} ||v_s||), \quad (2)$$

$$M_{klij} \equiv \partial \pi_{kl} / \partial u_{ij} \equiv \partial^2 w / \partial u_{ij} \partial u_{kl},$$

which clearly possesses the structure of hypoplasticity [1, 2], a state-of-the-art engineering model originally adopted because sand is incrementally nonlinear, and responds with different stress increases depending on whether the load is being increased ($v_{ij} > 0$, $||v_s|| > 0$) or decreased ($v_{ij} < 0$, $||v_s|| > 0$). It is reassuring to see that the realism of hypoplasticity is based on the elementary physics that granular deformation is depleted if the grains jiggle; and it is satisfying to realize that the complexity of plastic flows derives from the simplicity of stress relaxation.

Under cyclic loading of small amplitudes, because the shear rate is not constant, T_g oscillates and never has time to grow to its stationary value of $T_g \sim ||v_s||$. Therefore, the plastic term $u_{ij}/\tau \sim T_g$ remains small, and the system's behavior is rather more elastic than rendered by Eq (2).

The complete equation for u_{ij} is in fact somewhat more complex, [44]

$$d_t u_{ij} = (1 - \alpha) v_{ij} - u_{ij}^* / \tau - u_{\ell\ell} \delta_{ij} / \tau_1, \quad (3)$$

$$1/\tau = \lambda T_g, \quad 1/\tau_1 = \lambda_1 T_g, \quad (4)$$

where u_{ij}^* is the deviatoric (or traceless) part of u_{ij} and $d_t \equiv \partial_t + v_k \nabla_k$. The modifications are: (1) The relaxation time for u_{ij}^* and $u_{\ell\ell}$ are different. (2) A shear rate v_{ij} yields an elastic deformation rate $d_t u_{ij}$ that is smaller by the factor of $(1 - \alpha)$.

In contrast to strain relaxation $\sim u_{ij}/\tau$ that is irreversible, α accounts for reversible processes (such as rolling). Without relaxation, elastic and total strain are always proportional,

and for say $\alpha = 2/3$, u_{ij} is a third of ε_{ij} . Circumstances are then reversible and quite analogous to a solid – aside from the fact that one needs to move three times as far to achieve the same deformation. So the physics accounted for by α is akin to that of a lever. [This is also the reason why the stress, or counter-force, is smaller by the same factor, see Eq(19).] Note since any granular plastic motion such as rolling and slipping, be it reversible or irreversible, become successively improbable when the grains are less and less agitated, we expect

$$\alpha(T_g) \rightarrow 0, \quad \text{for } T_g \rightarrow 0, \quad (5)$$

implying granular media are fully elastic at vanishing granular temperature.

B. Mass, Entropy and Granular Entropy

The energy density $w_0(s, \rho)$ of a quiescent Newtonian fluid depend on the entropy density s and mass density ρ , both per unit volume. Defining the temperature and chemical potential as $T \equiv \partial w_0 / \partial s|_\rho$ and $\mu \equiv \partial w_0 / \partial \rho|_s$, we note that they can be computed only if the functional dependence of $w_0(s, \rho)$ is given. The pressure, a prominent quantity in fluid mechanics, is also a conjugate variable, as it is given by $P \equiv \partial \bar{w} / \partial v$ at constant sv , where $v \equiv 1/\rho$ is the specific volume, $\bar{w} \equiv w_0 v$ the energy per unit mass. Again, P is given once $w_0(s, \rho)$ is. (Note it is not independent from μ and T , since it may be written as $P = -w_0 + Ts + \mu\rho$.)

The conserved energy w depends also on the momentum density $g_i = \rho v_i$, and is generally given as $w = w_0 + g^2/2\rho$. So the complete set of variables is given as s, ρ and g_i , and the hydrodynamic theory of Newtonian fluids consists of five evolution equations for them. Being a structure of an actual theory, these equations contain w_0, P , also $T, \mu, v_i \equiv \partial w / \partial g_i$. They are closed only when w_0 is specified. [45]

In continuum-mechanical theories, the entropy s is not always given the attention it deserves. The basic facts underpinning its importance are: The conserved energy w is, in equilibrium, equally distributed among all degrees of freedom, macroscopic ones such as ρ, g_i , and microscopic ones such as electronic excitations or phonons (ie, short wave length sound waves). The entropy s is the macroscopic degree of freedom that subsumes all microscopic ones (typically of order 10^{23}), and accounts for the energy contained in them. Off equilibrium, energy is more concentrated in a few degrees of freedom, typically the macroscopic ones. The one-way, irreversible transfer of energy from the macroscopic to the microscopic ones

– in fluid mechanics from ρ, g_i to s – is what we call dissipation, and the basic cause for irreversibility. A proper account of dissipation must consider the variable s , its conjugate variable T , and the entropy production R [with R/T denoting the rate at which entropy is being increased, see Eq (9)]. This remains so for systems (such as granular media) that typically execute isothermal changes.

The energy density of a solid depends on an additional tensor variable, the elastic strain $u_{ij} = \varepsilon_{ij}$, which in crystals is very close to the total strain. The associated conjugate variable $\pi_{ij} \equiv -\partial w_0/\partial u_{ij}$ is the elastic stress – where linear elasticity, or $\pi_{ij} \sim u_{ij}$, represents the simplest case. The hydrodynamic theory of solids consists of eleven evolution equations, for the variables s, ρ, g_i, u_{ij} , which in their structure contain the conjugate variables T, μ, v_i, π_{ij} .

Displaying solid and liquid behavior, granular media have the same variables – in addition to the one that quantifies granular jiggling, for which a scalar should suffice if the motion is sufficiently random. We call it *granular entropy* s_g , and define it to contain all inter-granular degrees of freedom: the stochastic motion of the grains (in deviation from the smooth, macroscopic velocity) and the elastic deformation resulting from collisions. We divide all microscopic degrees of freedom contained in s into the [46] inner- and inter-granular ones, $s - s_g$ and s_g , with the conjugate variables $T \equiv \partial w_0/\partial(s - s_g)$ and $T_g \equiv \partial w_0/\partial s_g$. Equilibrium is established, when both temperatures are equal, and s_g vanishes. (There are overwhelmingly more inner than inter granular degrees of freedom. When all degrees have the same amount of energy, there is practically no energy left in s_g .) The equilibrium conditions are:

$$s_g = 0, \quad \bar{T}_g \equiv T_g - T = 0. \quad (6)$$

As zero is the value s_g invariably returns to if unperturbed, it is an energy minimum. Expanding the s_g -dependent part of the energy $w_2 \equiv w - w(s_g = 0)$, we take [47]

$$w_2(s, \rho, s_g) = s_g^2/(2\rho b), \quad \bar{T}_g \equiv \partial w_2/\partial s_g|_s = s_g/\rho b, \quad (7)$$

with $b(s, \rho) > 0$. So the twelve independent variables are: $s, s_g, \rho, g_i, u_{ij}$ and the hydrodynamic theory consists of evolution equations for them all, of which six are given by Eq (3). The rest will be given in section III. These equations will contain w_0 and the conjugate variables: $T, \bar{T}_g, \mu, v_i, \pi_{ij}$, also the pressure, given as

$$P_T \equiv -\partial \bar{w}_0/\partial v \equiv -w_0 + \mu\rho + sT + s_g\bar{T}_g, \quad (8)$$

with the derivative taken at constant $sv, s_g v$ and u_{ij} . As we shall see in Eq (24), this is the pressure that accounts for the contribution of agitated grains.

C. History Dependence and Fabric Anisotropy

Finally, some remarks about the special role of the density in granular behavior. First, it is quite independent of the compression $u_{\ell\ell}$: Plastic motion rearranges the packaging and change the density by up to 20%, without any elastic compression. Second, the local density only changes if there is some jiggling and agitation of the grains, $\bar{T}_g \neq 0$. Even when non-uniform, a given density remains forever if the grains are at rest. So, if a pouring procedure produces a density inhomogeneity, this will persist as long as the system is left unperturbed, providing an explanation for the history dependence of static stress distribution. Sometimes, these density inhomogeneities have a preferred direction, say, a density gradient along \hat{x} . With density-dependent elastic coefficients, the system will then mimic fabric anisotropy, displaying a stress-distribution reminiscent of an anisotropic medium – even when it consists of essentially round grains and the applied stress is isotropic. Our working hypothesis, given a preliminary validation in section IV A 3, is that both effects are covered by density inhomogeneities. The static stress of a sand pile is calculated there and compared to experiments for two densities, the first uniform and the second with a reduced core density, which we argue is a result of different pouring procedures, being rain-like and funnel-fed, respectively.

III. GRANULAR SOLID HYDRODYNAMICS (GSH)

This section presents the remaining six evolution equations. They will be explained but not derived, see [24] for more details and the complete derivation.

A. Entropy Production

The evolution equation for the entropy density s is

$$\partial_t s + \nabla_i (s v_i - \kappa \nabla_i T) = R/T, \quad (9)$$

$$\begin{aligned} R = & \eta v_{ij}^* v_{ij}^* + \zeta v_{\ell\ell}^2 + \kappa (\nabla_i T)^2 \\ & + \gamma \bar{T}_g^2 + \beta (\pi_{ij}^*)^2 + \beta_1 \pi_{\ell\ell}^2. \end{aligned} \quad (10)$$

Eq (9) is the balance equation for the entropy s . It is (with R unspecified) quite generally valid, certainly so for Newtonian fluids and solids. The term $s v_i$ is the convective one that accounts for the transport of entropy with the local velocity, and $\kappa \nabla_i T$ is the diffusive term that becomes operative in the presence of a temperature gradient. $R/T > 0$ is the source term. It vanishes in equilibrium, and is positive-definite off it, to account for the fact that the conserved energy w always goes from the macroscopic degrees of freedom to the microscopic ones, $w \rightarrow s$.

The functional dependence of R changes with the system. In liquids, R is fed by shear and compressional flows, and by temperature gradients [3], as depicted by the first line of Eq (10). In equilibrium, we have $v_{ij}, \nabla_i T = 0$; off it, the quadratic form with positive shear and compressional viscosity, $\eta, \zeta > 0$ and heat diffusion coefficient, $\kappa > 0$, ensures that the entropy s can only increase. In fact, the terms of the first line are, in an expansion of R , the lowest order positive ones that are compatible with isotropy.

The second line of Eq (10), with $\gamma, \beta, \beta_1 > 0$, displays the additional dissipative mechanisms relevant for granular media. As discussed in the introduction, a finite \bar{T}_g or π_{ij} , indicating some jiggling or deformation of the grains, will both relax and give rise to entropy production. Since granular stress π_{ij} will not dissipate for $\bar{T}_g = 0$, we require $\beta, \beta_1 \rightarrow 0$ for $\bar{T}_g \rightarrow 0$.

Being part of the total entropy, the granular entropy s_g obeys a rather similar equation, though it needs to account for a two-step irreversibility, $w \rightarrow s_g \rightarrow s$, the fact that the energy goes from the macroscopic degrees of freedom to the mesoscopic, inter granular ones of s_g , and from there to the microscopic, inner granular ones of s , never backwards,

$$\partial_t s_g + \nabla_i (s_g v_i - \kappa_g \nabla_i \bar{T}_g) = R_g/\bar{T}_g, \quad (11)$$

$$R_g = \eta_g v_{ij}^* v_{ij}^* + \zeta_g v_{\ell\ell}^2 + \kappa_g (\nabla_i \bar{T}_g)^2 - \gamma \bar{T}_g^2. \quad (12)$$

Eq (11) has the exact same form as Eq (9), so do the first three terms of R_g . But R_g also has a negative contribution. The three positive ones, with $\eta_g, \zeta_g, \kappa_g > 0$, account for $w \rightarrow s_g$, how shear and compressional flows, and gradients in the granular temperature produce s_g , the jiggling of the grains. The negative term $-\gamma \bar{T}_g^2$ accounts for $s_g \rightarrow s$, how the jiggling turns into heat. There is the same term, though with negative sign, in R , because the same amount of energy arriving at s must have left s_g . As emphasized, all transport coefficients $\eta, \eta_g, \zeta, \zeta_g, \kappa, \kappa_g, \gamma, \beta, \beta_1$ are functions of the state variables (which may alternatively be taken as $T, \bar{T}_g, \rho, \pi_{\ell\ell}$ and $\pi_s^2 \equiv \pi_{ij}^* \pi_{ij}^*$).

In the stationary and uniform limit, for $R_g = 0$ and $\nabla_i T_g = 0$, macroscopic flows produce the same amount of granular entropy as is leaving, implying

$$\gamma \bar{T}_g^2 = \eta_g \mathbf{v}_{ij}^* \mathbf{v}_{ij}^* + \zeta_g \mathbf{v}_{\ell\ell}^2. \quad (13)$$

This is the relation employed to arrive at Eq (2), showing that hypoplasticity holds in the limit of stationary shear rates. Given a shear rate, part of its energy will turn into s_g , which in turn will leak over to s . At the same time, some of the flow's energy will heat up the system directly, with the ratio of the two dissipative channels parameterized by η/η_g and ζ/ζ_g . In dry sand, η, ζ are probably negligible and shall be neglected below – though they should be quite a bit larger in sand saturated with water: A macroscopic shear flow of water implies much stronger microscopic ones in the fluid layers between the grains, and the dissipated energy contributes to R .

Finally, we consider the \bar{T}_g -dependence of η_g, ζ_g, γ . Expanding them,

$$\eta = \eta_0 + \eta_1 \bar{T}_g, \quad \zeta_g = \zeta_0 + \zeta_1 \bar{T}_g, \quad \gamma = \gamma_0 + \gamma_1 \bar{T}_g, \quad (14)$$

we shall assume $\eta_0, \zeta_0 = 0$, because

- R_g then stays well defined for $\bar{T}_g \rightarrow 0$, see Eq (12);
- Viscosities typically vanish with temperature;
- This fits the Bagnold scaling;
- For $\gamma_0 \gg \gamma_1 \bar{T}_g$ and $\gamma_0 \ll \gamma_1 \bar{T}_g$, respectively, we have from Eq (13), for $\mathbf{v}_{\ell\ell} = 0$,

$$\bar{T}_g = (\eta_1/\gamma_0) |\mathbf{v}_{ij}^*|^2, \quad \bar{T}_g = \sqrt{\eta_1/\gamma_1} |\mathbf{v}_{ij}^*|. \quad (15)$$

This ensures the existence of an elastic regime at vanishing \bar{T}_g , see section IV B 2.

B. Conservation Laws

The three evolution equations left to be specified are conservation laws, for mass, energy and momentum,

$$\partial_t \rho + \nabla_i(\rho v_i) = 0, \quad \partial_t w + \nabla_i Q_i = -\rho v_i \nabla_i \phi, \quad (16)$$

$$\partial_t(\rho v_i) + \nabla_i(\sigma_{ij} + \rho v_i v_j) = -\rho \nabla_i \phi, \quad (17)$$

where ϕ is the gravitational potential (on the earth surface, we have $-\nabla_i \phi = G_i$, the gravitational constant pointing downwards). Without specifying the fluxes Q_i, σ_{ij} , these equations are always valid, quite independent of the system, and express the simple fact that being locally conserved quantities (in the absence of gravitation), energy, momentum and mass obey continuity equations. The basic idea of the hydrodynamic theory is to require the structure of the fluxes Q_i, σ_{ij} to be such that, with the temporal derivatives of the variables given by Eqs (3,9,11,16,17), the thermodynamic relation

$$\begin{aligned} \partial_t w(s, s_g, \rho, g_i, u_{ij}) &= (\partial w / \partial s) \partial_t s + (\partial w / \partial s_g) \partial_t s_g + (\partial w / \partial \rho) \partial_t \rho \\ &\quad + (\partial w / \partial g_i) \partial_t g_i + (\partial w / \partial u_{ij}) \partial_t u_{ij} \\ &= T \partial_t s + \bar{T}_g \partial_t s_g + \mu \partial_t \rho + v_i \partial_t g_i - \pi_{ij} \partial_t u_{ij} \end{aligned}$$

is identically satisfied, irrespective of w 's functional form. This is a rather confining bit of information, enough to uniquely fix the two fluxes as

$$Q_i = (w + P_T) v_i + \sigma_{ij} v_j - \kappa T \nabla_i T - \kappa_g \bar{T}_g \nabla_i \bar{T}_g, \quad (18)$$

$$\sigma_{ij} = (1 - \alpha) \pi_{ij} + (P_T - \zeta_g v_{\ell\ell}) \delta_{ij} - \eta_g v_{ij}^*, \quad (19)$$

with P_T given by Eq (8), and v_{ij}^* being the deviatoric (or traceless) part of v_{ij} . (For details of derivation see [24].) Although now specified to fit granular physics as codified in Eqs (3,9,11), these are still fairly general results, valid irrespective what concrete form w assumes. Moreover, they also nicely demonstrate the dependence on the number and types of variables: Eliminating s_g , or equivalently, taking $\bar{T}_g = 0$, in Eqs (8,18,19), one obtains the solid hydrodynamics. [48] Further eliminating u_{ij} by taking $\pi_{ij} = 0$ leads to the fluid hydrodynamics.

Focusing on the plastic motion, the standard approach (especially the thermodynamic consideration by Houlsby and coworkers, [40]) employs the plastic strain $p_{ij} \equiv \varepsilon_{ij} - u_{ij}$ as

the independent variable. Although this starts from the same insight about plastic motion, the connection between elastic strain, stress and energy, so similar in solids and granular media, with formulas that hold for both systems, is lost – or at least too well hidden to be useful, see also the discussion in section IV C 1.

Enforcing a velocity gradient v_{ij} , the rate of work being received by the system is $-\sigma_{ij}v_{ij} = -[(1-\alpha)\pi_{ij} + P_T\delta_{ij}]v_{ij} + [\zeta_g v_{\ell\ell}v_{\ell\ell} + \eta_g v_{ij}^*v_{ij}^*]$, see Eq (18). Of these, the terms in the first square brackets, being proportional to the velocity and hence odd under time inversion, are reactive; while those $\sim v^2$ in the second bracket are even and dissipative. Work received via an odd term will leave if its sign is changed by inverting time's direction; work received via an even term stays, as happens only with dissipative processes. The reappearance of the same factor $(1-\alpha)$ as in Eq (3) is not an accident, but required by energy conservation. If the same velocity leads to an elastic deformation that is smaller by $(1-\alpha)$, then just as with a lever, the force counteracting this deformation $\sigma_{ij} = (1-\alpha)\pi_{ij} + \dots$ is smaller by the same factor.

This concludes the derivation of the structure of GSH, or granular solid hydrodynamics, given by Eqs (3,8), (9,10,11,12) and (16,17,18,19).

IV. VALIDATION OF GSH

The advantage of GSH is two-fold, its clear connection to the elementary granular physics as spelt out in the introduction, and more importantly, the stringency of its structure. It cannot be changed at will to fit experiments, without running into difficulties with general principles. The only remaining liberty is the choice of the functional dependence for the energy and some transport coefficients. As this implies much less wiggle room than with typical continuum-mechanical models, any agreement with experimental data is less designed, “hand-crafted,” and more convincing, especially with respect to the starting physics.

In what follows, we shall first examine granular statics, for a medium at rest, $T_g = 0$, then go on to granular dynamics, with enforced flows or stress changes, and some accompanying jiggling, $T_g \neq 0$. An expression for the conserved energy w will be proposed that, in spite of its relative simplicity, reproduces many important granular features when embedded into GSH. As discussed above Eq (6), we divide w into three parts: the micro-, macro- and

mesoscopic ones,

$$w = w_0(s, \rho) + [w_1(u_{ij}, \rho, g_i) + g^2/2\rho] + w_2(s_g, \rho). \quad (20)$$

The first [49] accounts for the inner-granular degrees of freedom, all subsumed as heat into the true entropy s . We take $w_0 = \langle E(s)/m \rangle \rho$, where $E(s)$ is the energy of a grain, m its mass, and $\langle \rangle$ denotes the average. The second consists of the contributions from the macroscopic variables of momentum density g_i and the elastic strain u_{ij} , where w_1 is given by Eq (21) below. The third, $w_2(s_g, \rho)$ of Eq (7), is further specified in section IV B 1. It accounts for the inter-granular degrees of freedom, the mesoscaled, strongly fluctuating elastic and kinetic contributions.

A. Granular Statics, $\mathbf{T}_g = \mathbf{T}$

Given an energy $w_1(u_{ij})$, we can use the stress $\pi_{ij}(u_{ij}) \equiv -\partial w_1/\partial u_{ij}$ and $u_{ij} = \frac{1}{2}(\nabla_i U_j + \nabla_j U_i)$ to close the stress balance $\nabla_j \pi_{ij}(r_i) = \rho G_i$, and determine $\pi_{ij}(r_i)$ with appropriate boundary conditions. As this is done without any knowledge of the plastic strain, we may with some justification call this *granular elasticity* [31].

The relation $u_{ij} = \frac{1}{2}(\nabla_i U_j + \nabla_j U_i)$ remains valid because of the following reasons: In an elastic medium, the stressed state is characterized by a displacement field from a unique reference state, in which the elastic energy vanishes. Because there is no plastic deformation U_i^p , the total displacement is equal to the elastic one. Circumstances appear at first quite different in granular media. Starting from a reference state, a stressed one is produced by the displacement $U_i + U_i^p$, with typically $U_i^p \gg U_i$. Due to sliding and rolling, U_i^p is highly discontinuous, but U_i remains slowly varying, because the cost in elastic energy would otherwise be prohibitive. Fortunately, U_i^p is quite irrelevant: We have innumerable reference states, all with vanishing elastic energy and connected to one another by purely plastic deformations. As a result, we can, for any given displacement $U_i + U_i^p$, switch to the reference state that is separated from the original one by U_i^p , and to the stressed one by U_i . Now, the circumstances are completely analogous to that of an elastic medium.

1. Yield Surfaces

An important aspect of granular behavior, in the space spanned by the variables, is the existence of yield surfaces. We take them to be the divide between two regions, one in which stable elastic solutions are possible, the other in which they are not – so the system must flow and cannot come to rest. A natural and efficient way to account for yield is to code it into the energy, a scalar. Given the stress balance, the energy is extremal [31] – minimal if convex and maximal if concave. Having the energy being convex within the yield surface, and concave beyond it, any elastic solution that is stable within the surface, will be eager to get rid of the excess energy and become unstable against infinitesimal perturbations beyond it.

2. The Elastic Energy w_1

Our present choice for the elastic energy is [24, 33–35],

$$w_1(\rho, u_{ij}) = \mathcal{B}\sqrt{\Delta} (2\Delta^2/5 + u_s^2/\xi), \quad (21)$$

where $\Delta \equiv -u_{\ell\ell}$, $u_s^2 \equiv u_{ij}^* u_{ij}^*$. The energy w_1 is convex only for $u_s/\Delta \leq \sqrt{2\xi}$, or equivalently $\pi_s/P_\Delta \leq \sqrt{2/\xi}$ (where $P_\Delta \equiv \frac{1}{3}\pi_{\ell\ell}$, $\pi_s^2 \equiv \pi_{ij}^* \pi_{ij}^*$), which coincides with the Drucker-Prager condition. [50] Taking $\xi = 5/3$ gives a friction angle of about 28° . We further take $\mathcal{B} = \mathcal{B}_0 \mathcal{B}_1(\rho) \mathcal{C}(\rho, u_{ij})$, where \mathcal{B}_0 is a constant, and

$$\mathcal{B}_1 = [(\rho - \rho_{\ell p}^*)/(\rho_{cp} - \rho)]^{0.15}, \quad (22)$$

$$2\mathcal{C} = 1 + \tanh[(\Delta_0 - \Delta)/\Delta_1]. \quad (23)$$

The coefficient \mathcal{B}_1 diverges for the “random closed-pack” density, ρ_{cp} , and is convex only between ρ_{cp} and the “random loose pack” density $\rho_{\ell p}$. [$\rho_{\ell p}^*$ is a constant chosen to yield the right value for $\rho_{\ell p}$ with the relation $\rho_{\ell p} \equiv (11\rho_{cp} + 9\rho_{\ell p}^*)/20$.] It accounts for (1) the lack of elastic solutions for $\rho < \rho_{\ell p}$, when the grains loose contact with one another; (2) the stiffening of granular elasticity with growing density, until it (as an approximation for becoming very large) diverges at ρ_{cp} .

With Δ_0, k_1, k_2, k_3 being constants, and $\Delta_0 = k_1\rho - k_2u_s^2 - k_3$, we have $\mathcal{C} = 1$ for $\Delta \ll \Delta_0$, and $\mathcal{C} = 0$ for $\Delta \gg \Delta_0$. It changes from 1 to 0 in a neighborhood of Δ_1 around Δ_0 , destroying

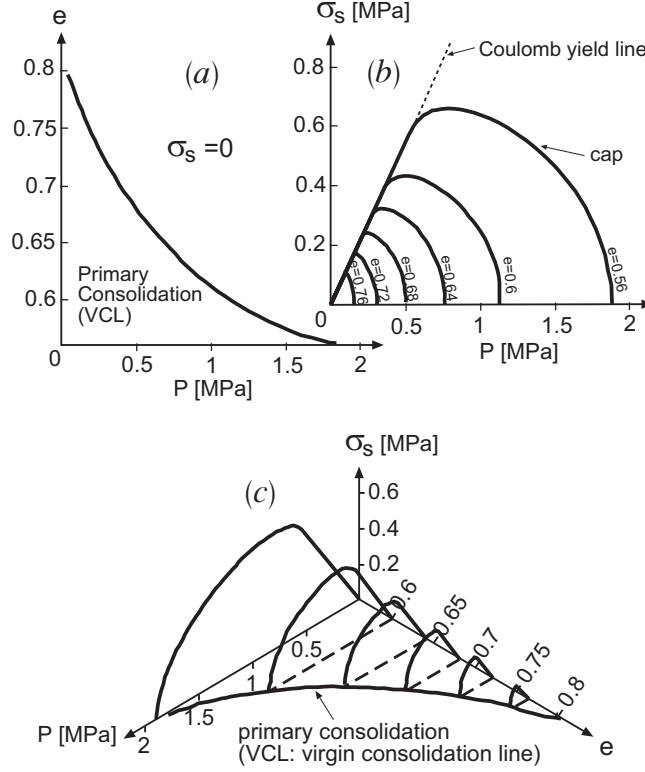


FIG. 1: Yield surfaces as coded in the energy of Eqs (21,22,23), a function of the pressure, shear stress, and void ratio. (a): The *virgin consolidation line*. (b): The bending of the Coulomb yield line, as a function of e . (c): Combination of (a) and (b).

the energy's convexity there. Taking Δ_0 to grow with the density and fall with u_s^2 limits the region of stable elastic solutions to sufficiently small Δ -values, reproducing the virgin consolidation curve and the so-called caps at varying void ratios e , see Fig 1.

3. Stress Distribution for Silos, Sand Piles and Point Loads

Three classic cases, a silo, a sand pile and a granular sheet under a point load, are solved employing the stress expression derived from the energy of Eq (21), producing rather satisfactory agreement with experiments.

Silos For tall silos, the classic approach is given by Janssen, who starts from the assumption that the ratio between the horizontal and vertical stress is constant, $k_J = \sigma_{rr}/\sigma_{zz}$. Assuming in addition that σ_{zz} only depends on z , not on r , Janssen finds the vertical stress σ_{zz} saturating exponentially with height – a result well verified by observation. (He leaves

σ_{rz} and all three radial components: $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ and $\sigma_{z\theta}$ undetermined.) Having calculated σ_{zz} , one needs the value of k_J to obtain σ_{rr} , usually provided by $k_J \simeq 1 - \sin \varphi$, with φ the friction angle measured in triaxial tests. This makes φ the only bulk material parameter in silo stress distributions. We shall refer to this as the Jaky formula, although it is also attributed to Kézdi. Being important for the structural stability of silos, this formula is (with a safety factor of 1.2) part of the construction industry standard, see eg. DIN 1055-6, 1987. We believe this formula goes well beyond its practical relevance, that it is a key to understanding granular stresses, because it demonstrates the intimate connection between stress distribution and yield, a connection that has not gained the wide attention it deserves. Starting from Eq (21), we calculated [32] all six components of the stress tensor, verifying the Janssen assumptions to within 1%, and found the Janssen constant k_J well rendered by the Jaky formula.

Point Loads The stress distribution at the bottom of a granular layer exposed to a point force at its top is calculated [32] employing Eq (21), without any fit parameter. Both vertical and oblique point forces were considered, and the results agree well with simulations and experiments using rain-like preparation. In addition, the stress distribution of a sheared granular layer exposed to the same point force is calculated and again found in agreement with experimental data, see [32] for more details and references.

Sand Piles The fact that the pressure distribution below sand piles and wedges, instead of always displaying a single central peak, may sometimes show a dip, has intrigued and fascinated many physicists, prodding them to think more carefully and deeply about sand. Recent experimental investigations established the following connection: A single peak results when the pile is formed by rain-like pouring from a fixed height; the dip appears when the pile is formed by funneling the grains onto the peak, from a shifting funnel always hovering slightly above the peak. Employing Eq (21) to consider the stress distribution in sand wedges, we found the pressure at the bottom of the pile to show a single central peak if a uniform density is assumed. The peak turns into a pressure dip, if density inhomogeneity, with the center being less compact, is assumed. The two calculated pressure distributions are remarkably similar to the measured ones, see [31]. The nonuniform density, we believe, is a consequence of pile formation using the hovering funnel: Since the funnel is always just above the peak, the grains are placed there with very little kinetic energy, resulting in a center region below the peak that has a low density. Those grains that do not find a stable

position roll down the slope and gather kinetic energy. When they crash to a stop at the flanks, they compact the surrounding, achieving a much higher density.

B. Granular Dynamics, $T_g \neq T$

If a granular medium is exposed either to stress changes, or a moving boundary, the grains will flow, displaying both a smooth, macroscopic velocity, $v_i \neq 0$, and some stochastic jiggling, $s_g \sim \bar{T}_g \neq 0$. Then the following effects will come into play: First, the energy is extended by a s_g -dependent contribution, $w_2(s_g, \rho)$, see Eq (7). Second, the transport coefficients of Eq (14) become finite. Most importantly, third, the relaxation times τ, τ_1 of Eq (3) are no longer infinite, implying the presence of plastic flows.

1. The s_g -Dependent Part of the Energy

Specifying the expansion coefficient $b(\rho)$ of Eq (7) as $b = b_0(1 - \rho/\rho_{cp})^a$, we find

$$P_T = a \rho b_0 \bar{T}_g^2 (1 - \rho/\rho_{cp})^{a-1} (\rho/2\rho_{cp}) \quad (24)$$

by employing Eq (8). The density dependence of the expansion coefficient $b(\rho)$ is chosen such that it reproduces the observed volume-dilating pressure contribution $P_T \sim f_2/(\rho_{cp} - \rho)$ from agitated grains [36–38]. However, we cannot take $a = 0$ as it would imply a diverging granular entropy s_g for $\rho \rightarrow \rho_{cp}$. Therefore, we take a to be positive but small, where $a \approx 0.1$ appears appropriate. (Note that with w_0/ρ independent of ρ and $w_1/\rho \sim \Delta^{2.5}$ – where Δ rarely exceeds 10^{-4} – the respective density derivative and pressure contribution is zero and negligibly small.)

2. The Hypoplastic Regime

We may choose our parameters such that \bar{T}_g is small at typical velocities of elasto-plastic deformations, though large enough to cover both limits of Eq (15). Then the first term of Eq (19) dominates, because all other terms ($\sim P_T, \eta_g, \zeta_g$) are of order \bar{T}_g^2 . Then we have $\partial_t \sigma_{ij} = (1 - \alpha) \partial_t \pi_{ij} = (1 - \alpha) M_{ijkl} \partial_t u_{kl}$, with $\partial_t u_{ij}$ given by Eq (3). Stress relaxation, the culprit producing irreversible plasticity, is a term $\sim \bar{T}_g$. For very slow shear flows and $\bar{T}_g \sim ||v_{ij}||^2$ [first of Eq (15)], it is quadratically small and negligible. This is the elastic regime. At

somewhat faster shear flows, the relation $\bar{T}_g \sim ||v_{ij}||$ [second of Eq (15)] renders $\partial_t \sigma_{ij}$ rate-independent, giving it the basic structure of hypoplasticity, Eq (3). Comparing this results to a state-of-the-art hypoplastic model, we found impressively quantitative agreement, see Fig 2. This is remarkable, because the anisotropy of these figures, determined essentially by M_{ijkl} , is a calculated quantity: $M_{ijkl} \equiv \partial^2 w_1 / \partial u_{ij} \partial u_{kl}$, with w_1 given by Eq (21).

3. The Butterfly Cycle

Our last example for validation is not a direct comparison of GHD to some experimental data, but rather an examination of what GHD does, unforced and uncrafted, under typical elasto-plastic deformations. It is solved numerically for stress paths in the triaxial geometry (ie. $\sigma_{xx} = \sigma_{yy}$, $\sigma_{ij} = 0$ for $i \neq j$, similarly for u_{ij}), including all energy terms given above, except \mathcal{C} of Eq (23) that is set to 1 (assuming the yield surface is sufficiently far away). All transport coefficients depend on T_g as specified, but are otherwise constant, independent of stress and density. Also, all variables are taken to be spatially uniform, reducing a set of partial differential equations to ordinary ones in time. In spite of these major simplifications,

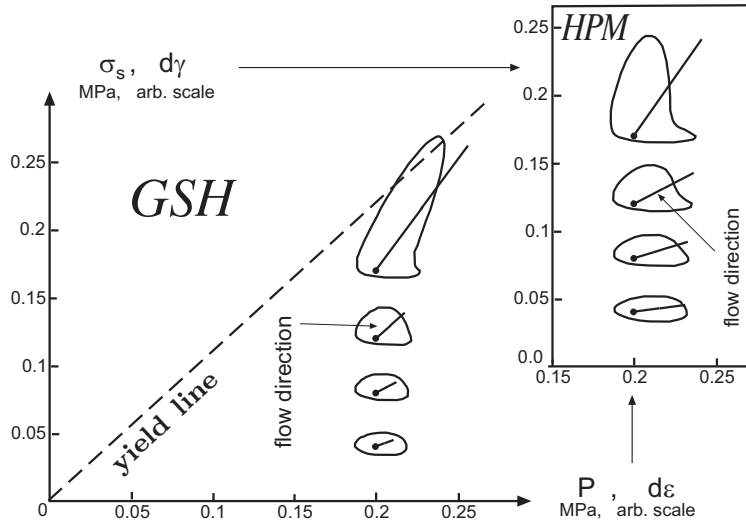


FIG. 2: The change in strain $d\gamma \equiv (v_{11} - v_{33})dt$, $d\varepsilon \equiv -(2v_{11} + v_{33})dt$ for given stress rate starting from different points in the stress space, spanned by σ_s, P , as calculated employing (1) GSM, the present theory (taking $1 - \alpha = 0.22$, $\tau/\tau_1 = 0.09$, $\zeta_g/\eta_g = 0.33$, $\lambda\sqrt{\eta_g/\gamma} = 114$), and (2) HPM, a typical hypoplastic model, see [39] for more figures and details.

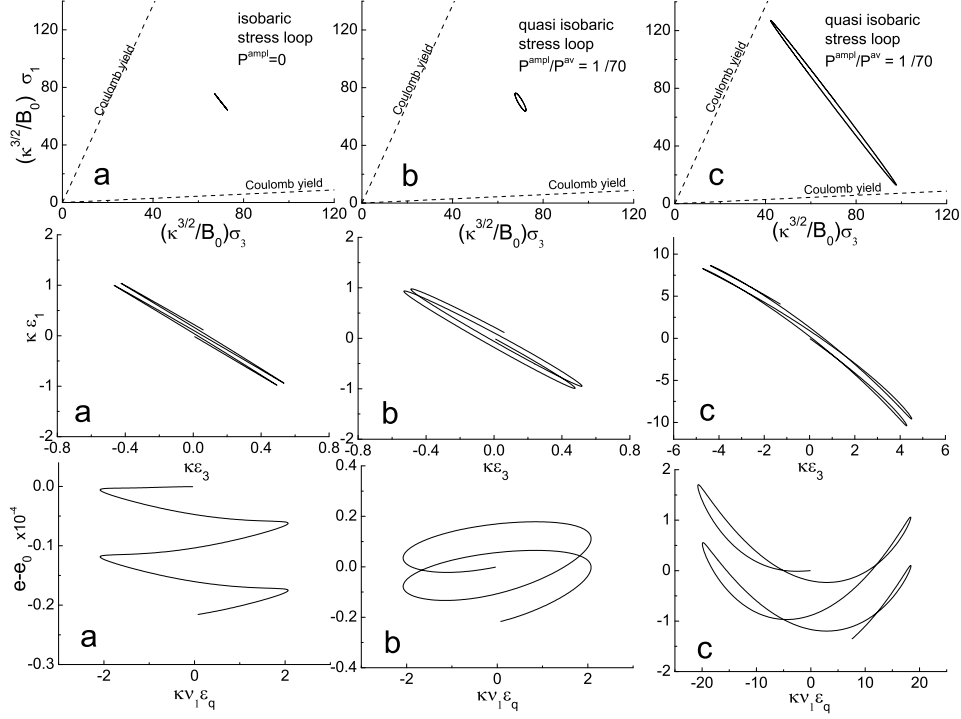


FIG. 3: Upper row: radial stress σ_1 versus axial stress σ_3 , rescaled by $B_0\kappa^{-3/2}$. Middle row: radial strain $\varepsilon_1 = \int v_{xx}dt$ versus axial strain $\varepsilon_3 = \int v_{zz}dt$. Lower row: $e - e_0$ (with e_0 the initial void ratio) versus shear strain $\varepsilon_q = \int (v_{zz} - v_{xx})dt$, rescaled by $\nu_1\kappa$. The stress loads are isobaric for (a) and quasi-isobaric for (b,c); the cyclic amplitude is small for (a,b) and large for (c). The associated strain loci and void ratio are: sawtooth-like for (a), coil-like for (b), butterfly-like (or double-looped) for (c). [The large-amplitude, isobaric plot is quite similar to (c).]

the results as rendered in Fig 3 display such uncanny realism that it seems obvious GSH has captured some important elements of granular physics. We consider a test with the stress given as

$$P = P^{av} + P^{ampl} \cos(2\pi ft), \quad q = q^{ampl} \cos(2\pi ft + \varphi). \quad (25)$$

Numerical solutions were computed for isobaric test with $P^{ampl} = 0$ (ie. $P = \text{constant}$) and quasi-isobaric test, with $P^{ampl} \ll P^{av}$ (ie. $P \approx \text{constant}$). The results are shown in Fig 3. they are obtained using the dimensionless parameters: $\kappa \equiv \sqrt{\zeta_1\gamma_1}/\rho b = 18257$, $(\gamma_0/\gamma_1)^2 (\rho_{cp} b^2 \kappa^{3/2}/\mathcal{B}_0 b_0) = 1.07 \times 10^{-6}$, $\lambda_1/\lambda = 0.09$, $\nu_1^2/2 \equiv \eta_1/3\zeta_1 = 1$, $\lambda\sqrt{\eta_1/\gamma_1} = 114$, $\alpha = 0$. The initial conditions are: $e_0 = 0.68085$ (or $\rho_0 = 0.94\rho_{cp}$), v_{ij} , \bar{T}_g , $\partial_t \bar{T}_g$, $\partial_t \rho$, $\partial_t u_{ij} = 0$. The averaged pressure $P^{av} \equiv \sigma_{ii}/3$ is $P^{av} = 70\mathcal{B}_0\kappa^{-3/2}$, and the amplitude $q^{ampl} \equiv \sigma_3 - \sigma_1$ is $5\nu_1\kappa^{3/2}q^{ampl}/6\mathcal{B}_0 = 10$ for (a,b) and 100 for (c,d). The frequency of

$P^{\text{ampl}}, q^{\text{ampl}}$ is $f = 12(\gamma_0/b\rho)$, and the phase lag between them is $\varphi = 58^\circ$.

C. Competing Concepts and Misconceptions

Finally, we revisit two previous approaches to come to terms with granular behavior, *granular thermodynamics* by Houlsby et al [40], and *granular statistical mechanics* by Edwards et al [41]. We shall compare GSH to both assuming at most superficial familiarity with them. Also, we refute some misconceptions that have become unfortunately widespread, especially the one about energy not being conserved in sand [*sic*]. These are at best a nuisance in exchanges with referees; and at worst actual obstacles in the progress of our coming to grips with granular modeling.

1. Granular Thermodynamics

Although considerable work and thoughts have gone into applying thermodynamics to granular media and plastic flow, especially from Houlsby and Collins [40], its basic points are clear and easy to grasp. Taking the entropy production as

$$R = \pi_{ij} \partial_t p_{ij} \quad (26)$$

(where p_{ij} denotes, as before, the plastic strain), it is obvious that the usual linear Onsager force-flux relation, $\partial_t p_{ij} \sim \pi_{ij}$, hence $R \sim \pi_{ij}^2$, does not give a rate-independent R . Therefore, Houlsby, Collins and coworkers consider instead

$$R = \sqrt{\chi_{ijkl} \partial_t p_{ij} \partial_t p_{kl}} = (\chi_{ijkl} \partial_t p_{ij} \partial_t p_{kl}) / \sqrt{\chi_{ijkl} \partial_t p_{ij} \partial_t p_{kl}},$$

a rate-independent expression. Equating it to Eq (26), with $\pi_{ij} = -\partial F / \partial p_{ij}$, and F being the free energy density, one then solves for the plastic strain p_{ij} with a given F . One example gives $\partial_t p_{ij} \neq 0$ on a yield surface, characterized by some components of π_{ij} being constant, and $\partial_t p_{ij} = 0$ off it.

GSH starts with the same R , but possesses the additional variable T_g , for which $\bar{T}_g \sim ||v_s||$ frequently holds, see Eq (13). The linear Onsager force-flux relation

$$\partial_t p_{ij} = \beta \pi_{ij} \quad \text{with} \quad \beta \sim \bar{T}_g, \quad (27)$$

therefore suffices to yield an rate-independent $R \sim \bar{T}_g \pi_{ij}^2$. Note Eq (27) leads directly to the relaxation term: Because $\partial_t u_{ij} + \partial_t p_{ij} = v_{ij}$, we have $\partial_t u_{ij} - v_{ij} = -\beta \pi_{ij} = -u_{ij}/\tau$, with $1/\tau \sim \bar{T}_g$. (The last equal sign holds because π_{ij}, β, τ are all functions of u_{ij} , with β, τ as yet unspecified.)

Summarizing, without the variable \bar{T}_g , Houlsby and Collins needed to go beyond the well-verified and -substantiated procedure of linear Onsager force-flux relation to maintain rate-independence, obtaining a plastic flow that is confined to the yield surface. In GSH, rate-independence arises naturally within the confines of linear Onsager relation, producing a plastic flow that is as realistic as hypoplasticity, and finite also off the yield surface.

2. Granular Statistical Mechanics

Generally speaking, it is important to remember that of all microscopic degrees of freedom, the inner-granular ones are many orders of magnitude more numerous than the inter-granular ones. It is the former that dominate the entropy and any entropic considerations. When revisiting granular statistical mechanics, especially the Edwards entropy, it is useful to keep this in mind.

Taking the entropy $S(E, V)$ as a function of the energy E and volume V , or $dS = (1/T)dE + (P/T)dV$, the authors of [41] argue that *a mechanically stable agglomerate of infinitely rigid grains at rest* has, irrespective of its volume, vanishing energy, $E \equiv 0, dE = 0$. The physics is clear: However we arrange these rigid grains that neither attract nor repel each other, the energy remains zero. Therefore, $dS = (P/T)dV$, or $dV = (T/P)dS \equiv X dS$. The entropy S is obtained by counting the number of possibilities to package grains for a given volume, and taking it to be e^S . Because a stable agglomerate is stuck in one single configuration, some tapping or similar disturbances are needed to enable the system to explore the phase space.

In GSH, the present theory, grains are neither infinitely rigid, nor always at rest, hence the energy contains both an elastic and a s_g -dependent contribution. [51] And the question is whether granular statistical mechanics is a legitimate limit of GSH. We are not sure, but a yes answer seems unlikely, as both are conceptually at odds in two points, the first more direct, the second quite fundamental: (1) Because of the Hertz-like contact between grains, very little material is being deformed at first, with the compressibility diverging at

vanishing compression. This is a geometric fact independent of how rigid the bulk material is. Infinite rigidity is therefore not a realistic limit for sand. (2) As emphasized, the number of possibilities to arrange grains for a given volume is vastly overwhelmed by the much more numerous configurations of the inner granular degrees of freedom, especially phonons. Maximal entropy S for given energy therefore realistically implies minimal macroscopic energy, such that a maximally possible amount of energy is in S (or heat), equally distributed among the inner-granular degrees of freedom. Maximal number of possibilities to package grains for a given volume is a very different criterion.

3. Energy Conservation

Stemming ultimately from a loose vocabulary, some alleged difficulties to model sand are based on fallacies that need to be refuted here.

The essential difference between granular gas and ideal (atomic or molecular) gas is that the particles of the first undergo non-elastic, dissipative collisions. As a result, their kinetic energy is not conserved, and the velocity distribution typically lacks the time to arrive at the equilibrium Gaussian form. Quantifying the kinetic energy as a granular temperature T_g , it is therefore hardly surprising that the fluctuation-dissipation theorem (FDT), formulated in terms of T_g , is frequently violated. These are sound results, obtained from a healthy but truncated model that takes the grains as the basic microscopic entity with no heat content. However, some of the further conclusions are deduced forgetting this simplification, rendering them patently absurd. These, and their *[refutation in italic]*, are listed below:

- As the energy is not conserved in sand, neither thermodynamics nor the hydrodynamic method are valid. *[Only the kinetic energy dissipates in granular media, not the total energy. The latter, including kinetic, elastic and heat contributions, remains conserved – as it is in any other system. And only the conservation of total energy is important for thermo- and hydrodynamics.]*
- FDT, along with other general principles either derived from it or in its conceptual vicinity (such as the Onsager reciprocity relation) are all violated. *[There are two versions of FDT, only the one given in terms of T_g is violated, not the one in terms of the true temperature T . The latter is a general principle and always valid. For*

instance, the volume fluctuation is given as $\langle \Delta V^2 \rangle = T(\partial^2 F / \partial V^2)^{-1}$, with F the associated free energy, for a copper block, a single grain, and a collection of grains. If the grains in the collection are jiggling, there is an extra contribution $\sim T_g^2$ in F , see Eq (7), that considerably increases the value of $\langle \Delta V^2 \rangle$. The Onsager relation remains valid because the true FDT holds.]

- The Onsager relation is also violated because the microscopic dynamics, the collision of the grains, is dissipative and hence irreversible. [The true microscopic dynamics is that in terms of atoms and molecules, the building blocks of the grains. Their dynamics is, as in any other system, reversible.]

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- [42] In the usual picture, force chains consisting of infinitely rigid grains is what sustains shear stresses. This does not contradict the above scenario, it is just a different description of the same circumstance: With contacts that are Hertzian (or Hertz-like), grains are infinitely soft at first contact, irrespective of how stiff the bulk material is, since very little material is then being deformed. There is therefore, realistically speaking, always some granular deformation and elastic energy present in any force chain. Now, because grains get rapidly stiffer when being compressed further, the displacement is small, and infinite rigidity is frequently a good approximation. Yet it is the loss of this tiny deformation by tapping that is the cause for the flattening of the pile.

- [43] “Jiggling” is used throughout, for random motion of the grains, large or small, that occurs when they are rearranging. Sometimes, words such as wiggle, creep, or crawl may be more appropriate.
- [44] even assuming hard grains, with an elastic strain u_{ij} that is typically tiny, of order 10^{-4}
- [45] Frequently, it is enough to know w_0 in a small environment around given values of s and ρ , or equivalently, of T and P , if these are taken as the independent variables.
- [46] Typical inner granular degrees of freedom are again phonons and electronic excitations.
- [47] With $dw_2 = Td(s - s_g) + T_g ds_g = Tds + \bar{T}_g ds_g$, we have $T_g \equiv \partial w_2 / \partial s_g|_{s-s_g}$ and $\bar{T}_g \equiv \partial w_2 / \partial s_g|_s$.
- [48] In solids, density change and compression are not usually independent. We may account for this by formally setting $P_T = 0$.
- [49] Assuming that only w_0 depends on s neglects effects such as thermal expansion which, however, can be easily included if needed.
- [50] Only if an energy expression depends on the third strain invariant, could it possibly contain an instability at the true Coulomb condition.
- [51] That grains neither attract nor repel each other is accounted for by the stress vanishing if s_g and u_{ij} do. Then $w_1, w_2 = 0$ and $w_0 \sim \rho$, see Eq (20), implying $\sigma_{ij} = \partial(w_0/\rho)/\partial(1/\rho)\delta_{ij} = 0$.